

Sequences and Series

- **Sequence:** A sequence is an arrangement of numbers in definite order according to some rule.

Also, we define a sequence as a function whose domain is the set of natural numbers or some subset of the type $\{1, 2, 3 \dots k\}$.

- A sequence containing finite number of terms is called a finite sequence.
- sequence containing infinite number of terms is called an infinite sequence.

- A general sequence can be written as

$$a_1, a_2, a_3 \dots a_{n-1}, a_n, \dots$$

Here, $a_1, a_2 \dots$ etc. are called the terms of the sequence and a_n is called the general term or n^{th} of the sequence.

- **Fibonacci sequence:** An arrangement of numbers such as 1, 2, 4, 6, 10 ... has no visible pattern. However, the sequence is generated by the recurrence relation given by

$$a_1 = 1, a_2 = 2, a_3 = 4$$

$$a_n = a_{n-2} + a_{n-1}, n > 3$$

This sequence is called the Fibonacci sequence.

- Let $a_1, a_2, \dots a_n, \dots$ be a given sequence. Accordingly, the sum of this sequence is given by the expression $a_1 + a_2 + \dots + a_n + \dots$

This is called the series associated with the given sequence.

The series is finite or infinite according as the given sequence.

A series is usually represented in a compact form using sigma notation (Σ).

This means the series $a_1 + a_2 + \dots + a_{n-1} + a_n \dots$ can be written as $\sum_{k=1}^n a_k$.

- **n^{th} term of an AP**

The n^{th} term (a_n) of an AP with first term a and common difference d is given by $a_n = a + (n - 1) d$.

Here, a_n is called the general term of the AP.

- **n^{th} term from the end of an AP**

The n^{th} term from the end of an AP with last term l and common difference d is given by $l - (n - 1) d$.

Example: Find the 12th term of the AP 5, 9, 13 ...

Solution: Here, $a = 5, d = 9 - 5 = 4, n = 12$

$$\begin{aligned}
 a_{12} &= a + (n - 1) d \\
 &= 5 + (12 - 1) 4 \\
 &= 5 + 11 \times 4 \\
 &= 5 + 44 \\
 &= 49
 \end{aligned}$$

- **Sum of n terms of an AP**

- The sum of the first n terms of an AP is given by $S_n = n[2a + (n-1)d]$, where a is the first term and d is the common difference.
- If there are only n terms in an AP, then $S_n = n[a + l]$, where $l = a_n$ is the last term.

Example : Find the value of $2 + 10 + 18 + \dots + 802$.

Solution: 2, 10, 18... 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be n terms in the series. Then,

$$a_n = 802$$

$$\Rightarrow a + (n - 1) d = 802$$

$$\Rightarrow 2 + (n - 1) 8 = 802$$

$$\Rightarrow 8(n - 1) = 800$$

$$\Rightarrow n - 1 = 100$$

$$\Rightarrow n = 101$$

$$\text{Thus, required sum} = n[2a + (n-1)d] = 101[2 + 800] = 40602$$

- **Properties of an Arithmetic progression**

- If a constant is added or subtracted or multiplied to each term of an A.P. then the resulting sequence is also an A.P.
- If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

- **Arithmetic mean**

- For any two numbers a and b , we can insert a number A between them such that a, A, b is an A.P. Such a number i.e., A is called the arithmetic mean (A.M) of numbers a and b and it is given by $A = \frac{a+b}{2}$.
- For any two given numbers a and b , we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

Let $A_1, A_2 \dots A_n$ be n numbers between a and b such that $a, A_1, A_2 \dots A_n, b$ is an A.P.

Here, common difference (d) is given by $\frac{b-a}{n+1}$.

Example: Insert three numbers between -2 and 18 such that the resulting sequence is an A.P.

Solution: Let A_1, A_2 , and A_3 be three numbers between -2 and 18 such that $-2, A_1, A_2, A_3, 18$ are in an A.P.

Here, $a = -2, b = 18, n = 5$

$$\therefore 18 = -2 + (5 - 1) d$$

$$\Rightarrow 20 = 4 d$$

$$\Rightarrow d = 5$$

$$\text{Thus, } A_1 = a + d = -2 + 5 = 3$$

$$A_2 = a + 2d = -2 + 10 = 8$$

$$A_3 = a + 3d = -2 + 15 = 13$$

Hence, the required three numbers between -2 and 18 are $3, 8$, and 13 .

- **Geometric Progression:** A sequence is said to be a geometric progression (G.P.) if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by r .
- In standard form, the G.P. is written as $a, ar, ar^2 \dots$ where, a is the first term and r is the common ratio.
- **General Term of a G.P.:** The n^{th} term (or general term) of a G.P. is given by $a_n = ar^{n-1}$

Example: Find the number of terms in G.P. $5, 20, 80 \dots 5120$.

Solution: Let the number of terms be n .

Here $a = 5, r = 4$ and $t_n = 5120$

$$n^{\text{th}} \text{ term of G.P.} = ar^{n-1}$$

$$\therefore 5(4)^{n-1} = 5120$$

$$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$$

$$\Rightarrow (2)^{2n-2} = (2)^{10}$$

$$\Rightarrow 2n - 2 = 10$$

$$\Rightarrow 2n = 12$$

$$\therefore n = 6$$

- **Sum of n Term of a G.P.:** The sum of n terms (S_n) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & \text{if } r < 1 \\ \frac{a(r^n-1)}{r-1}, & \text{if } r > 1 \\ na, & \text{if } r = 1 \end{cases}$$

Example: Find the sum of the series $1 + 3 + 9 + 27 + \dots$ to 10 terms.

Solution: The sequence $1, 3, 9, 27, \dots$ is a G.P.

Here, $a = 1, r = 3$.

Sum of n terms of G.P. = $\frac{a(r^n - 1)}{r - 1}$ [$r > 1$]

$S_{10} = 1 + 3 + 9 + 27 + \dots$ to 10 terms

$$= \frac{1 \times [(3)^{10} - 1]}{(3 - 1)}$$

$$= \frac{59049 - 1}{2}$$

$$= \frac{59048}{2}$$

$$= 29524$$

- Three consecutive terms can be taken as ar, a, ar . Here, common ratio is r .
- Four consecutive terms can be taken as ar^3, ar, ar, ar^3 . Here, common ratio is r^2 .
- **Geometric Mean:** For any two positive numbers a and b , we can insert a number G between them such that a, G, b is a G.P. Such a number i.e., G is called a geometric mean (G.M.) and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \dots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, \dots, G_n, b$ is a G.P., then $G_1, G_2, \dots, G_n$ are given by $G_1 = ar, G_2 = ar^2, \dots, G_n = ar^n$

Where, r is calculated from the relation $b = ar^{n+1}$, that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

Example: Insert three geometric means between 2 and 162.

Solution: Let G_1, G_2, G_3 be 3 G.M.'s between 2 and 162.

Therefore, 2, $G_1, G_2, G_3, 162$ are in G.P.

Let r be the common ratio of G.P.

Here, $a = 2, b = 162$ and $n = 3$

$$r = \left(\frac{162}{2}\right)^{\frac{1}{3+1}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$$

$$G_1 = ar = 2 \times 3 = 6$$

$$G_2 = ar^2 = 2 \times (3)^2 = 2 \times 9 = 18$$

$$G_3 = ar^3 = 2 \times (3)^3 = 2 \times 27 = 54$$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$ and G.M. of two given positive real numbers a and b . Accordingly,

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$

- **Sum of n -terms of some special series:**

Sum of first n natural numbers

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

- Sum of squares of the first n natural numbers

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Sum of cubes of the first n natural numbers

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Example: Find the sum of n terms of the series whose n^{th} term is $n(n+1)(n-2)$.

Solution: It is given that

$$\begin{aligned} a_n &= n(n+1)(n-2) \\ &= n(n^2 + n - 2n - 2) \\ &= n(n^2 - n - 2) \\ &= n^3 - n^2 - 2n \end{aligned}$$

Thus, the sum of n terms is given by

$$\begin{aligned}
S_n &= \sum_{k=1}^n k^3 - \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k \\
&= \left[\frac{n(n+1)}{2} \right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2} \\
&= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} - \frac{2n+1}{3} - 2 \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n(n+1) - 2(2n+1) - 12}{6} \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n^2 + 3n - 4n - 2 - 12}{6} \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n^2 - n - 14}{6} \right] \\
&= \frac{n(n+1)(3n^2 - n - 14)}{12} \\
&= \frac{n(n+1)(3n^2 - 7n + 6n - 14)}{12} \\
&= \frac{n(n+1)[n(3n-7) + 2(3n-7)]}{12} \\
&= \frac{n(n+1)(n+2)(3n-7)}{12}
\end{aligned}$$